MATH 245 S20, Exam 1 Solutions

1. Carefully define the following terms: even, floor, Double Negation Theorem.
   We call integer n even if there exists an integer m with $n = 2m$. Let $x \in \mathbb{R}$. We call integer a the floor of x if $a \le x < a+1$. The Double Negation Theorem states: Let p be a proposition. Then $\neg\neg p \equiv p$.

2. Carefully define the following terms: Trivial Proof Theorem, Contrapositive Proof Theorem, converse
   The Trivial Proof Theorem says that for propositions p, q, we have $q \vdash p \to q$. The Contrapositive Proof Theorem says that for propositions p, q, if $\neg q \vdash \neg p$ is valid, then $p \to q$ is true. The converse of conditional proposition $p \to q$ is $q \to p$.

3. Let a, b, c be integers, with $a|b$ and $a|c$. Prove that $a|(b+c)$.

   Because $a|b$, there exists some integer s with $b = as$. Because $a|c$, there exists some integer t with $c = at$. Adding, we get $b + c = as + at = a(s+t)$. Because $s+t$ is an integer, $a|(b+c)$.

4. Let $m, n \in \mathbb{Z}$ with $m \ge n \ge 0$. Prove that $\binom{m}{n} = \binom{m}{m-n}$.

   We have $\binom{m}{n} = \frac{m!}{n!(m-n)!} = \frac{m!}{(m-n)!n!} = \frac{m!}{(m-n)!(m-(m-n))!} = \binom{m}{m-n}$.

5. Use truth tables to prove that $\neg(p \vee q) \equiv (\neg p) \wedge (\neg q)$.

   The 4th and 7th columns agree in the truth table at right.

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$(\neg p) \wedge (\neg q)$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

6. Let $x \in \mathbb{R}$. Prove that if 6 is irrational, then x is irrational.

   6 is rational since $6 = \frac{6}{1}$, and $6, 1$ are both integers. Hence "6 is irrational" is false, so the implication is vacuously true. (see Thm. 3.7b, if you'd like more details)

7. Prove or disprove: $\forall x \in \mathbb{Z},\ x + 1 > x$.
   The statement is true. Let $x \in \mathbb{Z}$ be arbitrary.

   Proof 1: Direct proof. Because $(x+1) - x = 1 \in \mathbb{N}_0$, we know $x + 1 \ge x$. But also $x + 1 \ne x$. Hence $x + 1 > x$ (by definition of $>$).
   Proof 2: Use a theorem. We know $1 > 0$ by our entry point. We also know that $x \ge x$ since $x - x = 0 \in \mathbb{N}_0$. We can combine using a theorem from the book (Thm 1.11) to get $x + 1 > x + 0 = x$.

8. Let p, q, r, s be propositions. Simplify $(p \to q) \to (r \to s)$ to use only $\vee, \wedge, \neg$ where only basic propositions are negated.

   Step 1: Using Conditional Interpretation three times, our proposition is equivalent to $(s \vee \neg r) \vee \neg(q \vee \neg p)$.
   Step 2: Using De Morgan's Law, this is equivalent to $(s \vee \neg r) \vee ((\neg q) \wedge (\neg\neg p))$.
   Step 3: Using Double Negation, this is equivalent to $(s \vee \neg r) \vee ((\neg q) \wedge p)$.

9. State Modus Ponens and prove it using other theorems (without truth tables).
   Theorem: Let p, q be propositions. Then $p, p \to q \vdash q$.
   Pf 1: We assume $p, p \to q$. By conditional interpretation, $q \vee \neg p$. By double negation, $\neg\neg p$. By disjunctive syllogism, q.
   Pf 2: We assume $p, p \to q$. We have $p \to q \equiv (\neg q) \to (\neg p)$, its contrapositive. By double negation, $\neg\neg p$. By modus tollens, $\neg\neg q$. By double negation again, q.

10. Prove or disprove: $\forall x \in \mathbb{R},\ \exists y \in \mathbb{R},\ \exists z \in \mathbb{R},\quad y^2 \le x^2 < z^2$.
    The statement is true. Let $x \in \mathbb{R}$ be arbitrary. Set $y = x, z = \sqrt{x^2+1}$. We calculate $y^2 = x^2$, so $y^2 \le x^2$. We also calculate $z^2 = (\sqrt{x^2+1})^2 = x^2 + 1 > x^2$. Hence $y^2 \le x^2 < z^2$. Note: Other choices of y, z are possible.